

Cretan($4t+1$) Matrices

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Abstract

A *Cretan*($4t+1$) matrix, of order $4t+1$, is an orthogonal matrix whose elements have moduli ≤ 1 . The only *Cretan*($4t+1$) matrices previously published are for orders 5, 9, 13, 17 and 37. This paper gives infinitely many new *Cretan*($4t+1$) matrices constructed using *regular Hadamard* matrices, *SBIBD*($4t+1, k, \lambda$), weighing matrices, generalized Hadamard matrices and the Kronecker product. We introduce an inequality for the radius and give a construction for a Cretan matrix for every order $n \geq 3$.

Keywords: *Hadamard matrices; regular Hadamard matrices; orthogonal matrices; symmetric balanced incomplete block designs (SBIBD); Cretan matrices; weighing matrices; generalized Hadamard matrices; 05B20.*

1 Introduction

An application in image processing (compression, masking) led to the search for orthogonal matrices, all of whose elements have modulus ≤ 1 and which have maximal or high determinant.

Cretan matrices were first discussed, per se, during a conference in Crete in 2014. This paper follows closely the joint work of N. A. Balonin, Jennifer Seberry and M. B. Sergeev [1, 2, 3].

The orders $4t$ (Hadamard), $4t-1$ (Mersenne), $4t-2$ (Weighing) are discussed in [4, 5, 6]. This present work emphasizes the $4t+1$ (Fermat type) orders with real elements ≤ 1 . Cretan matrices which are complex, based on the roots of unity or are just required to have at least one 1 are mentioned.

1.1 Preliminary Definitions

The absolute value of the determinant of any matrix is not altered by 1) interchanging any two rows, 2) interchanging any two columns, and/or 3) multiplying

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any row/or column by -1 . These equivalence operations are called *Hadamard equivalence operations*. So the absolute value of the determinant of any matrix is not altered by the use of Hadamard equivalence operation.

Write I_n for the identity matrix of order n , J for the matrix of all 1's and let ω be a constant. An *orthogonal* matrix, S , of order n , is square, has real entries and satisfies $SS^T = \omega I_n$. The *core* of a matrix is formed by removing the first row and column.

A *Cretan* matrix, S , of order n has entries with modulus ≤ 1 and at least one 1 per row and column. It satisfies $SS^T = \omega I_n$ and so it is an orthogonal matrix. A *Cretan*($n; \tau; \omega$) matrix, or *CM*($n; \tau; \omega$) has τ levels or values for its entries [1].

An *Hadamard matrix* of order n has entries ± 1 and satisfies $HH^T = nI_n$ for $n = 1, 2, 4t$, $t > 0$ an integer. Any Hadamard matrix can be put into *normalized form*, that is having the first row and column all plus 1s using Hadamard equivalence operations: that is it can be written with a core. A *regular Hadamard matrix* of order $4m^2$ has $2m^2 \pm m$ elements 1 and $2m^2 \mp m$ elements -1 in each row and column (see [7, 8]).

Hadamard matrices and weighing matrices are well known orthogonal matrices. We refer to [2, 9, 7, 10, 8] for more details and other definitions. The reader is pointed to [11, 12, 13] for details of generalized Hadamard matrices, Butson Hadamard matrices and generalized weighing matrices.

For the purposes of this paper we will consider an *SBIBD*(v, k, λ), B , to be a $v \times v$ matrix, with entries 0 and 1, k ones per row and column, and the inner product of distinct pairs of rows and/or columns to be λ . This is called the *incidence matrix* of the SBIBD. For these matrices $\lambda(v-1) = k(k-1)$, $BB^T = (k-\lambda)I + \lambda J$, and $\det B = k(k-\lambda)^{\frac{v-1}{2}}$.

For every *SBIBD*(v, k, λ) there is a complementary *SBIBD*($v, v-k, v-2k+\lambda$). One can be made from the other by interchanging the 0's of one with the 1's of the other. The usual use *SBIBD* convention that $v > 2k$ and $k > 2\lambda$ is followed.

We now define our important concepts the *orthogonality equation*, the *radius equation*(s), the *characteristic equation*(s) and the *weight* of our matrices.

Definition 1 (Orthogonality equation, radius equation(s), characteristic equation(s), weight). Consider the matrix $S = (s_{ij})$ comprising the variables x_1, x_2, \dots, x_τ .

The *matrix orthogonality equation*

$$S^T S = SS^T = \omega I_n \quad (1)$$

yields two types of equations: the n equations which arise from taking the inner product of each row/column with itself (which leads to the diagonal elements of ωI_n being ω) are called *radius equation*(s), $g(x_1, x_2, \dots, x_\tau) = \omega$, and the $n^2 - n$ equations, $f(x_1, x_2, \dots, x_\tau) = 0$, which arise from taking inner products of distinct rows of S (which leads to the zero off diagonal elements of ωI_n are called *characteristic equation*(s). Cretan matrices must satisfy the three equations: the orthogonality equation (1), the radius equation and the characteristic equation(s).

Notation: We use $\text{CM}(n; \tau; \omega; \det(\text{optional}); (t_1, t_2, \dots, t_\tau))$, or just $\text{CM}(n; \tau; \omega)$, where t_1, t_2, \dots, t_τ are the possible values (or levels) of the elements in CM.

1.2 Inequalities

Some inequalities are known for matrices which have real entries ≤ 1 . Hadamard matrices, $H = (H_{ij})$, which are orthogonal and with entries ± 1 satisfy the equality of Hadamard's inequality (2) [9]

$$\det(HH^\top) \leq \prod_{i=1}^n \sum_{j=1}^n |h_{ij}|^2, \quad (2)$$

have determinant $\leq n^{\frac{n}{2}}$. Further Barba [14] showed that for matrices, B , of order n whose entries are ± 1 ,

$$\det B \leq \sqrt{2n-1}(n-1)^{\frac{n-1}{2}} \text{ or asymptotically } \approx 0.858(n)^{\frac{n}{2}}. \quad (3)$$

For $n = 9$ Barba's inequality gives $\det B \leq \sqrt{17} \times 8^4 = 16888.24$. The Hadamard inequality gives 19683 for the bound on the determinant of the ± 1 matrix of order 9. So the Barba bound is better for odd orders. We thank Professor Christos Koukouvinos for pointing out to us that the literature, see Ehlich and Zeller, [15], yields a ± 1 matrix of order 9 with determinant 14336. These bounds have not been met for $n = 9$.

Koukouvinos also pointed out that in Raghavarao [16] a ± 1 matrix of order 13 with determinant 14929920 $\approx 1.49 \times 10^7$ is given. This is the same value given for $n = 13$ given by Barba's inequality. The Hadamard inequality gives 1.74×10^7 for the bound on the determinant of the ± 1 matrix of order 13.

These bounds have been significantly improved by Brent and Osborn [17] to give $\leq (n+1)^{\frac{(n-1)}{2}}$.

Wojtas [18] showed that for matrices, B , whose entries are ± 1 , of order $n \equiv 2 \pmod{4}$ we have

$$\det B \leq 2(n-1)(n-2)^{\frac{n-2}{2}} \text{ or asymptotically } \approx 0.736(n)^{\frac{n}{2}}. \quad (4)$$

This gives a determinant bound ≤ 73728 for order 10 whereas the weighing matrix of order 10 has determinant $9^5 = 59049$.

We observe that the determinant of a $\text{CM}(n; \tau; \omega; \det)$ is always $\omega^{\frac{n}{2}}$.

Hence we can rewrite the known inequalities of this subsection noting that only the Hadamard inequality applies generally for elements with modulus ≤ 1 . Thus we have

Theorem 1. Hadamard-Cretan Inequality *The radius of a Cretan matrix of order n is $\leq n$.*

2 Two Trivial Cretan(n) Families

The next two families are included for completeness.

2.1 The Basic Family

Lemma 1. Consider $C = aJ + b(J - I)$ of order n , a, b variables. This gives a $CM(n; 2; 1 + \frac{4(n-1)}{(n-2)^2})$ matrix of order n ie a $CM(n; 2; 1 + \frac{4(n-1)}{(n-2)^2}; \det; (1, \frac{-2}{n-2}))$.

Proof. Writing C with a on the diagonal and other elements b , the radius and characteristic equations become

$$a^2 + (n-1)b^2 = \omega \quad \text{and} \quad 2a + (n-2)b = 0.$$

Hence with $a = 1$ and $b = \frac{-2}{n-2}$ we have $\omega = 1 + \frac{4(n-1)}{(n-2)^2}$ for the required $CM(n)$ matrix. \square

Remark 1. For $n = 7, 9, 11, 13$ this gives $\omega = 1\frac{24}{25}, 1\frac{32}{49}, 1\frac{40}{81}$ and $1\frac{48}{121}$ respectively. These determinants are very small. However they do give a $CM(n; 2)$ for all integers $n > 0$.

2.2 Known Families

The following results may be found in [19] and [6].

Proposition 1. [Cretan(4t)] There is a $Cretan(4t; 2; 4t)$ for every integer $4t$ for which there exists an Hadamard matrix.

Proposition 2. [Cretan(4t-1)] There are $Cretan(4t-1; 2; \omega)$, $\omega = 4t+1 - \sqrt{t}$ and $\omega = \frac{2t^3+t-2t(2t-1)\sqrt{t}}{(t-1)^2}$ for every integer $4t$ for which there exists an Hadamard matrix.

The next two results are easy for the knowledgeable reader and merely mentioned here.

Proposition 3. [Cretan(4t-2)] There are $Cretan(4t-2; 3; k)$ whenever there is a $W(4t-2, k)$ weighing matrix. For $k = 4t-3$, the sum of two squares, and a $4W(4t-2, 4t-3)$ is known, the complex Cretan matrix $CM(4t-2; 3; 4t-2)$ has elements $i = \sqrt{-1}, 1$ or -1 .

Proposition 4. [Cretan(np)] There are complex $Cretan(np; p; n)$, whenever there exists a generalized Hadamard matrix based on the p th roots of unity.

2.3 The Additive Families

We will illustrate this construction using two Cretan matrices to give a Cretan matrix whose order is the sum of their orders. This shows how many possible matrices we might find for any n but again all the determinants are small.

Lemma 2. Let A and B be $CM(n_1; 3; \omega_1)$ and $CM(n_2; 3; \omega_2)$ respectively. Then $A \oplus B$ given by

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

is a $CM(n_1 + n_2; 4; \omega)$ matrix of order $n_1 + n_2$ with $\omega = \min(\omega_1, \omega_2)$. (Note it does not have one 1 per row and column.)

Remark 2. We note using smaller $CM(n_i; \tau; \omega_i)$ gives many inequivalent $CM(n; \tau; \omega)$ for any order $n = \sum_i n_i$, but the elements of all but the smallest submatrix will not contribute 1 to the resulting Cretan matrix.

Now with $n = n_1 + n_2$ for $21 = 4 + 17, 5 + 16, 6 + 15, 7 + 14, 8 + 13, 9 + 12, 10 + 11$ plus other combinations, the submatrices of orders n_1 and n_2 contribute differently to τ and ω . This means

Proposition 5. *There is a $Cretan(n; \tau; \omega)$ for every integer n .*

In Section 3.3 we explore the same Proposition 5 for more interesting τ .

3 Constructions for $Cretan(4t + 1; \tau)$ Matrices

We now describe a number of constructions for $Cretan(4t + 1)$ matrices.

3.1 Constructions using SBIBD

3.1.1 2-level $Cretan(4t + 1)$ matrices via $SBIBD(v = 4t + 1, k, \lambda)$

The following Theorem is a special case of the construction for 2-level $Cretan(v = 4t + 1)$ given in [6]. It also yields a valid $CM(37; 2)$.

Theorem 2. [6] *Let S be a $CM(v = 4t + 1; 2; \omega; (a, b))$ based on $SBIBD(v = 4t + 1, k, \lambda)$ then $a = 1$, $b = \frac{(k-\lambda) \pm \sqrt{k-\lambda}}{v-2k+\lambda}$ and $\omega = ka^2 + (v-k)b^2$, provided $|b| \leq 1$.*

Example 1. Using the La Jolla Repository <http://www.ccrwest.org/ds.html> of difference sets that Marshall Hall Jr found an $SBIBD(37, 9, 2)$. Using Theorem ?? we obtain $CM(37; 2; 12.325; (1, 0.345))$ and $CM(37; 2; 9.485; (1, 0.132))$. The complementary $SBIBD(37, 28, 21)$ does not give any Cretan matrix as $|b| \geq 1$.

We especially note the $(45, 12, 3)$ difference set, where the occurrence of the $Cretan(45, 2; 20\frac{1}{4})$ matrix and the $Cretan(45, 2; 14\frac{1}{16})$ matrices both arise from the $SBIBD(45, 12, 3)$: the complementary $SBIBD(45, 33, 24)$ does not yield any Cretan matrix.

Example 2. Orthogonal matrices of orders 13 and 21 may be constructed by using the $SBIBD(13, 4, 1)$ and $SBIBD(21, 5, 1)$ given in [20]. $CM(13; 2; 9.60; (1, \frac{3 \pm \sqrt{3}}{6}))$ and $CM(21; 2; 10; (1, -\frac{1}{6}))$ are given in Figure 1.

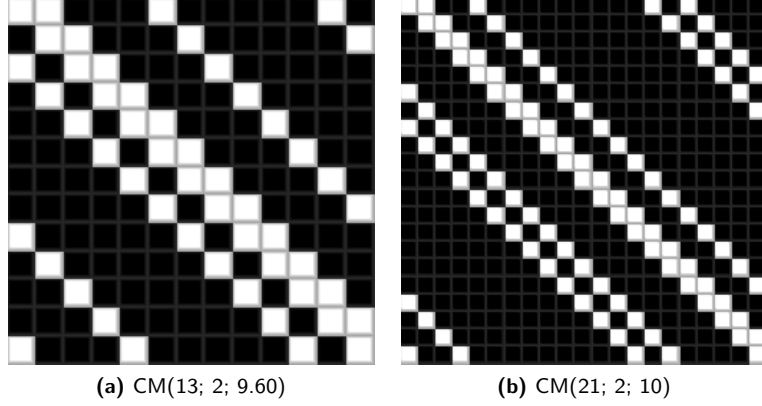


Figure 1: 2-level Cretan matrices of order 13 and 21

All the examples of $SBIBD(4t+1, k, \lambda)$ that we have given from the La Jolla Repository have been constructed using difference sets. Most of those we give arise from Singer difference sets and finite geometries: these $SBIBD((p^{n+1} - 1)/(p - 1), (p^n - 1)/(p - 1), (p^{n-1} - 1)/(p - 1))$ difference sets are denoted as $PG(n, p)$. The bi-quadratic type constructions are due to Marshall Hall [21]. There are many $SBIBD$ constructed without using difference sets.

3.1.2 Bordered Constructions

We do not elaborate on the next theorem here but note it gives many Cretan matrices $CM(v+1)$.

Theorem 3. *The matrix C below can be used to construct many $CM(v+; \tau; \omega)$ with borders by replacing the matrix B by an $SBIBD(v, k, \lambda)$.*

When a matrix C is written in the following form

$$C = \begin{bmatrix} x & s & \dots & s \\ s & & & \\ \vdots & & B & \\ s & & & \end{bmatrix}$$

B is said to be the *core* of C and the s 's are the *borders* of B in C . C is said to be in *bordered form*. The variables x are s can be realized in the cases described below.

3.1.3 Using Regular Hadamard Matrices

For details and constructions many of the known Regular Hadamard Matrices the interested reader is referred to [8, 7, 22].

Lemma 3. Let M be a regular Hadamard matrix of order $4m^2$ with $2m^2 + m$ positive elements per row and column. Then forming C as follows

$$C = \begin{bmatrix} 1 & s & \dots & s \\ s & & & \\ \vdots & & \frac{1}{2m}M & \\ s & & & \end{bmatrix}$$

gives a Cretan($4n^2 + 1; 4; 1$) matrix or $CM(4m^2 + 1; 4; 1; (0, 1, \frac{1}{2m}, \frac{-1}{2m}))$.

Proof. For C to be a Cretan matrix it must satisfy the orthogonality, radius and characteristic equations. These are

$$CC^T = (1 + 4m^2s^2)I_{4m^2+1} = (s^2 + 4m^2)I_{4m^2+1} = \omega I_{4m^2+1}$$

for the orthogonality equation, giving $s = 0$, $\omega = 1$ for the radius equation and 0 for the characteristic equations.

Hence we have a matrix of order $4m^2 + 1$ with elements $0, 1, \pm \frac{1}{2m}$ satisfying the required Cretan equations. \square

Corollary 1. Since there exists a regular (symmetric) Hadamard matrix of order $4 = 2^2$, $4^2 = 2^{2^2}$, $4^{4^2} = 2^{2^{2^2}}$..., there is a Cretan($n = 2^{2^{2^2}} \dots + 1; 4; 1$) for n a Fermat number.

Proof. Let S be the regular symmetric Hadamard matrix of order 4. Then the Kronecker product

$$S \times S \times \dots \times$$

is the required core for the construction in Lemma 3. \square

Example 3. Purported examples of pure Fermat matrices in Figure 2 for orders 5 and 17: levels a , b are white and black colours, the border level s is given in grey. However the reader is cautioned that though the figures appear to be Cretan matrices they are not. They are based on SBIBD, including the regular Hadamard matrix SBIBD($4m^2, 2m^\pm m, m^\pm m$) and require $c = a$. We note though that when $c = a \neq 1$ the radius and characteristic equations do not give meaningful real solutions.

Example 4. See Figure 3 for examples of a regular Hadamard matrix of order 36 and a purported new Balonin-Seberry type of 3-level Cretan(37) with complex entries that is a orthogonal matrix of order 37. A real Cretan(37; 2) does exist from Theorem 2 above (see example). \square

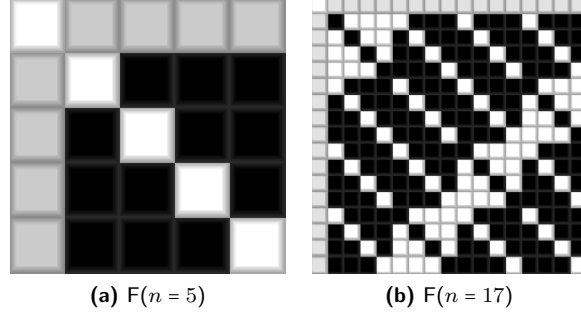


Figure 2: Orthogonal Cretan(Fermat) matrices for Fermat numbers 5 and 17

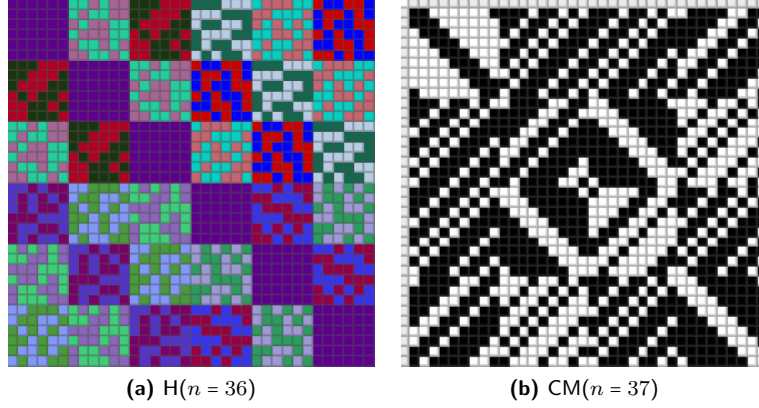


Figure 3: Regular Hadamard matrix of order 36 and a 3-level Cretan(37)

3.2 Using Normalized Weighing Matrix Cores

This next construction is not valid in the real numbers. However we can allow Cretan matrices to have complex elements and choose the diagonal to be $i = \sqrt{-1}$.

Lemma 4. *Suppose there exists a normalized conference matrix, B , of order $4t + 2$, that is a $W(4t + 2, 4t + 1)$. Then B may be written as*

$$B = \begin{bmatrix} i & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & F & \\ 1 & & & \end{bmatrix}.$$

This is a Cretan matrix.

Removing the first row and column of B to study the core F is unproductive.

3.2.1 Generalized Hadamard Matrices and Generalized Weighing Matrices

We first note that the matrices we study here have elements from groups, abelian and non-abelian, (see [11, 12, 13, 23, 24] for more information) and may be written in additive or multiplicative notation. The matrices may have real elements, elements $\in \{1, -1\}$, elements $|n| \leq 1$, elements $\in \{1, i, i^2 = -1\}$, elements $\in \{1, i, -1, -i, i^2 = -1\}$, integer elements $\in \{a + ib, i^2 = -1\}$, n th roots of unity, the quaternions $\{1$ and $i, j, k, i^2 = j^2 = k^2 = -1, ijk = -1\}$, $(a + ib) + j(c + id)$, a, b, c, d , integer and i, j, k quaternions or otherwise as specified.

We use the notations B^T for the transpose of G , B^H for the group transpose, B^C for the complex conjugate of B^T , B^Q for the quaternion conjugate and B^V for the quaternion conjugate transpose.

In all of these matrices the inner product of distinct rows a and b is $a \cdot b$ or $a \cdot b^{-1}$ depending on whether the group is written in additive or multiplicative form.

- **Generalized orthogonality:** A *generalized Hadamard matrix*, or *difference matrix*, $GH(gn, g)$ over a group of order g has the inner product of distinct rows the whole group the same number of n times. The inner product is $\{g_{i1}g_{j1}^{-1}, \dots, g_{in}g_{jn}^{-1}\}$. For example

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a & b & ab \\ 1 & b & ab & a \\ 1 & ab & a & b \end{bmatrix}; \quad GG^H = (\text{group})I_4 = (Z_2 \times Z_2)I$$

orthogonality is because of the definition of the inner product.

- **Butson Hadamard matrix** [11]

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}; \quad BB^C = 3I_3, \quad w^3 = 1, \quad 1 + w + w^2 = 0$$

is said to be a Butson Hadamard matrix. Orthogonality depends on the fact that the n th roots of unity add to zero.

- A *generalized/generalized Hadamard matrix* [11, 12, 13], $GH(np, G)$, where G is a group of order p , can also be written in additive form for example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix} \text{ is a } GH(6, Z_3).$$

- A generalized weighing matrix, $W = GW(np, G, k)$ [23], where G is a group of order p , has w non-zero elements in each column and W is orthogonal over G . The following two matrices are additive and multiplicative $GW(5, Z_3)$, respectively.

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 2 & 0 \\ 0 & 1 & * & 0 & 2 \\ 0 & 2 & 0 & * & 1 \\ 0 & 0 & 2 & 1 & * \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & w & w^2 & 1 \\ 1 & w & 0 & 1 & w^2 \\ 1 & w^2 & 1 & 0 & w \\ 1 & 1 & w^2 & w & 0 \end{bmatrix}.$$

* is zero but not the zero of the group.

Theorem 4. Any generalized Hadamard matrix or generalized weighing matrix is a $CM(n; g)$ over the group G , of order g , which may be the roots of unity.

3.3 The Kronecker Product of Cretan Matrices

Lemma 5. Suppose A and B are $CM(n_1; \tau_1; \omega_1)$ and $CM(n_2; \tau_2; \omega_2)$ then the Kronecker product of A and B written $A \times B$ is a $CM(n_1 n_2; \tau; \omega_1 \omega_2)$ where τ depends on τ_1 and τ_2 .

Example 5. From [6, 25] we see that $CM(3; 2; 2.25)$, $CM(7; 2; 5.03)$ and $CM(7; 2; 3.34)$ exist so there exist $CM(21; 3; 11.32)$ and $CM(21; 3; 7.52)$.

The Hadamard-Cretan bound gives, for $n = 21$, radius ≤ 21 .

From Balonin and Seberry [6] we have that since an SBIBD($p^r, \frac{p^r-1}{2}, \frac{p^r-3}{4}$) exists for all prime powers $p^r \equiv 3 \pmod{4}$ there exist $CM(p^r; 2; \omega)$ for all these prime powers (see Proposition 2). Hence using Kronecker products in the previous theorem and writing n as a product of prime powers we have

Theorem 5. There exists a $CM(n; \tau; \omega)$ $\omega > 1$ for all odd orders n , $n = \prod \rho \times p^{i_1} p^{i_2} \dots$, where ρ is an order for which a Cretan $CM(\rho = 4t + 1)$ is known and p^{i_1}, p^{i_2}, \dots are any prime powers $\equiv 3 \pmod{4}$, for some τ and ω .

Table 1 gives the present integers for which ρ is known. Similar theorems can be obtained for all even n .

Remark 3. We note that τ depends on the actual construction used. Combining $CM(n_1; 2; \omega_1 : (a, b))$ and $CM(n_2; 2; \omega_1 : (a, b))$ gives $CM(n_1 n_2; 3; \omega_{12} : (a^2, ab, b^2))$. General formulae for τ from CM with different levels are left as an exercise.

4 The Difference between Cretan($4t + 1; \tau$) Matrices and Fermat Matrices

The first few pure Fermat numbers are $v = 3, 5, 17, 257, 65537, 4294967297, \dots$. We note these are all $\equiv 1 \pmod{4}$ and may be constructed using Corollary 1. Figure 4 gives an early example of a Fermat matrix.



Figure 4: Core of Russian Fermat Matrix from mathscinet.ru

Finding 3-level orthogonal matrices of order $\equiv 1 \pmod{4}$ for non-pure Fermat numbers has proved challenging. Orders $n = 9$ and $n = 13$ are given in citej.

Orders $v = 2^{\text{even}} + 1$ called Fermat type matrices, pose an interesting class to study.

Orders $4t + 1$, t is odd, are $Cretan(4t + 1)$ - matrices; their order is neither a Fermat number ($2 + 1 = 3$, $2^2 + 1 = 4 + 1$, $2^{2^2} + 1 = 16 + 1$, $2^{2^{2^2}} + 1 = 256 + 1$, ...) nor a Fermat type number ($2^{\text{even}} + 1$). Examples of regular Hadamard matrices of order 36, giving the first $CM(37; 3; 1)$ matrix of order 37 [3] where 37 is not a Fermat number or Fermat type number, have been placed at site [26]. They use regular Hadamard matrices as a core and have the same, as any other Hadamard matrix, level functions. We call them $Cretan(4t + 1)$ matrices and will consider them further in our future work.

Matrices of the $Cretan(4t + 1)$ family made from Singer difference sets (see [21] also have orders belonging to the set of numbers $4t + 1$, t odd: these are different from the three-level matrices of Balonin-Sergeev (Fermat) family [27, 19] with orders $4t + 1$, t is 1 or even.

5 Summary

In this paper we have given new constructions for $CM(4t + 1)$. These are summarised in Table 1 for $4t + 1 < 200$.

Table 1: Some Cretan $CM(4t+1)$, $3 \leq 4t+1 \leq 199$

From Regular Hadamard Matrices ($\omega = 1$)				5	17	37	45	65
				101	145	197		
From Difference Sets (ds)								
v	k	λ	Existence	Difference set		Comment		
13	4	1	All Known	PG(2,3)		Unique Hall [28]		
21	5	1	All Known	PG(2,4)		Unique Hall [28]		
37	9	2	Exists	Biquadratic residue ds		Hall [28]		
45	12	3	All Known			La Jolla [20]		
57	8	1	All Known	PG(2,7)		Unique Hall [28]		
73	9	1	All Known	PG(2,8)		Unique Hall [28]		
85	21	5	Exists	PG(3,4)		[20]		
101	25	6	Exists	Biquadratic residue ds		Hall [28]		
109	28	7	Exists	Biquadratic residue ds		Hall [28]		
121	40	13	Exists	PG(4,3)		[20]		
133	33	8	Exists			La Jolla [20]		
197	49	12	Exists	Biquadratic residue ds		Hall [28]		
Kronecker Product			All Orders which are the Product a Known Order and of Prime Power $\equiv 3 \pmod{4}$					

Table 2: Cretan 2-level and 3-level $CM(4t \pm 1)$, $3 \leq 4t + 1 \leq 199$

v	$Method$	v	$Method$	v	$Method$
3	BM[4]+Prop:2	5	BM[4]	7	BM+Prop:2
9	BM[4]	11	BM[4]+Prop:2	13	BM[4]
15	Kronecker	17		19	Prop:2
21	from SBIBD Table:1	23	Prop:2	25	Kronecker
27	Prop:2	29		31	Prop:2
33	Kronecker	35	Kronecker	37	
39	Kronecker	41		43	Prop:2
45	from SBIBD Table:1	47	Prop:2	49	Kronecker
51		53		55	Kronecker
57	from SBIBD Table:1	59	Prop:2	61	
63	Kronecker	65	Kronecker	67	Prop:2
69	Kronecker	71	Prop:2	73	from SBIBD Table:1
75	Kronecker	77	Kronecker	79	Prop:2
81	Prop:2	83		85	from SBIBD Table:1
87		89		91	Kronecker
93	Kronecker	95	Kronecker	97	
99	Kronecker	101	from SBIBD Table:1	103	Prop:2
105	Kronecker	107	Prop:2	109	from SBIBD Table:1
111		113		115	Kronecker
117	Kronecker	119		121	from SBIBD Table:1
123		125	Kronecker	127	Prop:2
129	Kronecker	131	Prop:2	133	from SBIBD Table:1
135	Kronecker	137		139	Prop:2
141	Kronecker	143		145	
147	Kronecker	149		151	Prop:2
153		155	Kronecker	157	
159		161	Kronecker	163	Prop:2
165	Kronecker	167	Prop:2	169	Kronecker
171	Prop:2	173		175	Kronecker
177	Kronecker	179	Prop:2	181	
183		185		187	
189	Kronecker	191	Prop:2	193	
195	Prop:2	197	from SBIBD Table:1	199	Prop:2

6 Conclusions

Cretan matrices are a very new area of study. They have many research lines open: what is the minimum number of variables that can be used; what are the determinants and radii that can be found for $Cretan(n; \tau)$ matrices; why do the congruence classes of the orders make such a difference to the proliferation of Cretan matrices for a given order; find the Cretan matrix with maximum and minimum determinant for a given order; can one be found with fewer levels?

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